PSoC Today Presents:

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impedance meter

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Math. This is full of math. Math related to calculating complex impedance from the magnitude and phase information of voltage and current. These are the equations I used in my impedance meter project to determine the inductance and capacitance.

This document is a companion to the SynchDetectExp file, which explains synchronous detection in detail. Synchronous detection is used to measure the magnitude and phase of the voltage and current.

Details:

Measuring resistance is pretty straightforward process. Apply a voltage or current and measure the resulting instantaneous current or voltage. The relationship to resistance is R = V / I.

Measuring impedance is a more involved process since it describes a components response to a time-varying signal. For instance, a capacitor will provide more impedance for a signal at a low frequency and less impedance for a signal at a high frequency. To further complicate matters, the phase relationship between the current and voltage is affected by the impedance. A pure capacitance will cause the current to lead the voltage by 90 degrees, while a pure inductance will cause the current to lag the voltage by 90 degrees. Impedance is usually represented as a complex number with both a magnitude and phase. Magnitude represents the “resistance” of the component, and the phase describes the components effect on the phase relationship of the current through the component when compared to the voltage. These values change as a function of frequency, making impedance a tricky thing.

With the goal of ending up with an impedance value, how exactly do we derive it from a component?

Basics of complex impedance measurement:

We will begin with the following deceptively simple equation. This is, in a sense, a more pure version of Ohm's law:

Equation 1: $\tilde{V} = \tilde{I} * \tilde{Z}$

Equation 1 describes the relationship of a complex voltage (with magnitude and phase) to a complex current and impedance. Lets say that we could drive a voltage V into the component, and measure the resulting current I. Lets then say that the voltage V is a sinusoidal signal with a particular magnitude abs(V) represented as $|V|$ and phase angle(V) represented as $\angle\theta_V$. We can represent that voltage with a phasor $\tilde{V}$, which is a complex number, having both magnitude and phase.

If we were able to measure $\tilde{V}$ and $\tilde{I}$, we could determine $\tilde{Z}$ using Equation 2.

Equation 2: $\tilde{Z} = \frac{\tilde{V}}{\tilde{I}}$

We can further break down the relationship of complex values into their magnitude and impedance. This simplifies the calculation of $\tilde{Z}$ from $\tilde{V}$ and $\tilde{I}$.

Equation 3: $\tilde{Z} = \frac{|V|\angle\theta_V}{|I|\angle\theta_I}$

Equation 4: $|Z| = \left|\frac{|V|\angle\theta_V}{|I|\angle\theta_I}\right|$

Equation 5: $|Z| = \frac{|V|}{|I|}$

Equation 6: $\theta_Z = \angle\left(\frac{|V|\angle\theta_V}{|I|\angle\theta_I}\right)$

Equation 7: $\theta_Z = \theta_V - \theta I$

In the ideal world, we know that complex impedances are made up of purely real resistances, and purely imaginary reactance. Energy storage devices (Inductors and Capacitors) have a frequency dependent reactance, denoted in a series circuit as X_s.

Equation 8: $\tilde{Z} = R_s + jX_s$

Equation 9: $R_s + jX_s = \frac{|V|\angle\theta_V}{|I|\angle\theta_I}$

We can extract the purely real resistance

Equation 10: $R_s = real\left(\frac{|V|\angle\theta_V}{|I|\angle\theta_I}\right)$

Equation 11: $R_s = real\left(\frac{|V|}{|I|}\angle\theta_V - \theta_I\right)$

Equation 12: $R_s = \frac{|V|}{|I|}\cos(\theta_V - \theta_I)$

Equation 13: $R_s = |Z|\cos(\theta_Z)$

As well as the purely imaginary reactance

Equation 14: $X_s = imag(\frac{|V|\angle\theta_V}{|I|\angle\theta_I})$

Equation 15: $X_s = imag\left(\frac{|V|}{|I|}\angle\theta_V - \theta_I\right)$

Equation 16: $X_s = \frac{|V|}{|I|}\sin(\theta_V - \theta_I)$

Equation 17: $X_s = |Z|\sin(\theta_Z)$

From the series resistance and reactance, we can also determine the quality factor and the dissipation factor

Equation 18: $Q = \frac{X_s}{R_s}$

Equation 19: $D = \frac{1}{Q}$

The equivalent parallel combinations can be determined using the following transformations:

Equation 20: $L_p = L_s(1 + D^2)$

Equation 21: $C_p = \frac{C_s}{1 + D^2}$

Equation 22: $R_p = R_s\left(1 + \frac{1}{D^2}\right)$

So now that we have the math to calculate all the necessary values from the magnitude and phase relationship of the voltage and current ($|Z|, \theta_Z, R_s, X_s, L_s, C_s, R_p, L_p, C_p, Q, D$), how do we actually get the magnitude and phase values for the voltage and the current?

A brute force method for determining the complex impedance of a device would be to:

1. Drive the impedance with a sine wave of a known frequency and sample the drive voltages at a sufficiently fast sample rate (>> 2x excitation frequency)
2. Sample the current induced in the impedance with a at a sufficiently fast sample rate (>> 2x excitation frequency)
3. Compare the magnitude and phase difference between the voltage and current by computing the FFT of the sampled waveforms by extracting the magnitude and phase at the excitation frequency from the FFTs

This method requires fast simultaneous sampling ADC's and plenty of CPU horsepower to efficiently calculate the FFT. If fast simultaneous sampling ADC's or CPU horsepower are not available, there are alternative methods to determining the magnitude and phase relationships of waveforms using a bit more processing in the analog domain.

Enter, Synchronous Detection (the other document).

I realized while writing this that I have forgotten a lot of EE201. The Agilent Impedance measurement handbook is a fantastic document for covering the basics of impedance, as well as the nitty-gritty details of the really grungy stuff you have to worry about when making a capacitor that works out to a few 100 MHz. Look it up, its free online.

Appendix A:

Examples:

Equation 23: $X_c = \frac{1}{j\omega C}$

With $R_s = 0$, and $\theta_V = 0^\circ$

A 1 uF capacitor at 15.9 Hz will have the following impedance:

$$\tilde{Z} = X_c = \frac{1}{j\omega C} = \frac{1}{j2\pi 15.9 * 1e-6} = \frac{10e3}{j} = -j10e3$$

$$\tilde{V} = \tilde{I} * \tilde{Z} \rightarrow \frac{\tilde{V}}{\tilde{Z}} = \tilde{I} \rightarrow \frac{V\angle 0^\circ}{\tilde{Z}} = \tilde{I} \rightarrow \frac{V\angle 0^\circ}{-j * 10e3} = \tilde{I} \rightarrow \frac{V\angle 0^\circ}{10e3\angle -90^\circ} = \tilde{I} \rightarrow \frac{V}{10e3}\angle 0^\circ + 90^\circ$$

$$\tilde{I} = \frac{V}{10e3}\angle 90^\circ$$

The current through the capacitor will lead the voltage by a factor of 90 degrees. If the current is leading the voltage, then we have a capacitive circuit in the simple 2 element case.

Equation 24: $X_l = j\omega L$

With $R_s = 0$, and $\theta_V = 0^\circ$

A 1 mH capacitor at 159 Hz will have the following impedance:

$$\tilde{Z} = X_l = j\omega L = j2\pi 159 * 1e-3 = j$$

$$\tilde{V} = \tilde{I} * \tilde{Z} \rightarrow \frac{\tilde{V}}{\tilde{Z}} = \tilde{I} \rightarrow \frac{V\angle 0^\circ}{\tilde{Z}} = \tilde{I} \rightarrow \frac{V\angle 0^\circ}{j} = \tilde{I} \rightarrow \frac{V\angle 0^\circ}{1\angle 90^\circ} = \tilde{I} \rightarrow V\angle 0^\circ - 90^\circ$$

$$\tilde{I} = V\angle -90^\circ$$

The current through the inductor will lag the voltage by a factor of 90 degrees. If the current is lagging the voltage, then the circuit is inductive.

We can then calculate the equivalent series capacitance or inductance using the following equations:

Equation 25: $L_s = \frac{X_s}{\omega}$

Equation 26: $C_s = \frac{-1}{\omega X_s}$